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# On the generalized Laguerre polynomials of arbitrary (fractional) orders and quantum mechanics

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**Abstract.** The generalized Laguerre polynomials $Ł_\alpha^\beta(x)$ of arbitrary order $\alpha \in \mathbb{R}$ have been defined by the author (El-Sayed 1997 *Math. Sci. Res. Hot-line* **1** 7–14, 1999 to appear). In the latter reference it is proved that they are continuous as functions of $\alpha$, $\alpha \in \mathbb{R}$, and some other properties that generalize (interpolate) those of the classical Laguerre polynomials $L_n^\beta(x)$, $n = 1, 2, \ldots$ have been proved. Here we prove that $Ł_\alpha^\beta(x)$, $\alpha \in \mathbb{R}$ are orthogonal in $L_2(0, \infty)$ and are particular solutions of the differential equation

$$xD^2u(x) + (1 + \beta - x)Du(x) + \alpha u(x) = 0$$

generalizing the one for $L_n^\beta(x)$, $n = 1, 2, \ldots$. Also some applications in quantum mechanics are discussed.

## 1. Introduction

Let $L_1 = L_1[a, b]$, $0 \leqslant a < b < \infty$ be the class of Lebesgue integrable functions on the interval $[a, b]$.

**Definition 1.1.** *Let $f(x) \in L_1$, $\beta \in \mathbb{R}^+$. The Riemann–Liouville fractional (arbitrary) order integral of the function $f(x)$ of order $\beta$ is defined by (see [18, 19, 21])*

$$I_a^\beta f(x) = \int_a^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, \mathrm{d}s \tag{1}$$

*and when $a = 0$, we can write $I_0^\beta f(x) = f(x) * \phi_\beta(x)$, where $\phi_\beta(x) = x^{\beta-1}/\Gamma(\beta)$, for $x > 0$, $\phi_\beta(x) = 0$, for $x \leqslant 0$, and $\phi_\beta \to \delta(x)$ (the delta function), as $\beta \to 0$ (see [15]), and hence $I_0^\beta f(x) \to f(x)$, as $\beta \to 0$.*

Now the following lemma can be easily proved.

**Lemma 1.1.** *Let $\beta$ and $\gamma \in \mathbb{R}^+$. Then we have*

*(a) $I_a^\beta : L_1 \to L_1$, and if $f(x) \in L_1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.*
*(b) $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$, $n = 1, 2, 3, \ldots$ uniformly.* ∎

**Example 1.** *Let $\alpha \in (0, 1]$ and $\nu > -1$, then we have*

$$I_a^\alpha (x-a)^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)} (x-a)^{\nu+\alpha}.$$

*Also*

$$\lim_{\alpha\to 1} I_a^\alpha (x-a)^\nu = \frac{(x-a)^{\nu+1}}{\nu+1} = \int_a^t (s-a)^\nu \, \mathrm{d}s$$

*and*

$$\lim_{\alpha\to 0} I_a^\alpha (x-a)^\nu = (x-a)^\nu.$$

For the fractional order derivative we have mainly the following two definitions.

**Definition 1.2.** *The Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ of $f(x)$ is given by (see [18, 19, 21])*

$$\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \frac{\mathrm{d}}{\mathrm{d}x} I_a^{1-\alpha} f(x). \tag{2}$$

**Definition 1.3.** *The (Caputo [1] and El-Sayed approach [2–14, 19]) fractional derivative $D^\alpha$ of order $\alpha \in (0,1]$ of the absolutely continuous function $f(x)$ is given by*

$$D_a^\alpha f(x) = I_a^{1-\alpha} D f(x) \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{3}$$

*This definition is more convenient in many applications in physics, engineering and applied sciences. Moreover, it generalizes (interpolates) the definition of the integer-order derivative. The following lemma can be directly proved.*

**Lemma 1.2.** *Let $\alpha \in (0,1]$. If $f(x)$ is absolutely continuous on $[a,b]$, then*

*(a)* $D_a^\alpha f(x) = \mathrm{d}^\alpha f(x)/\mathrm{d}x^\alpha + [(x-a)^{-\alpha}/\Gamma(1-\alpha)] f(a)$.
*(b)* $\lim_{\alpha\to 1} D_a^\alpha f(x) = Df(x) \neq \lim_{\alpha\to 1} \mathrm{d}^\alpha f(x)/\mathrm{d}x^\alpha$.
*(c) If $f(x) = k$, $k$ is a constant, then $D_a^\alpha k = 0$, but* $\mathrm{d}^\alpha k/\mathrm{d}x^\alpha \neq 0$. ∎

**Example 2.** *Let $\alpha \in (0,1]$ and $\nu > 0$, then we have*

$$D_a^\alpha (x-a)^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} (x-a)^{\nu-\alpha}$$

*and*

$$\lim_{\alpha\to 1} D_a^\alpha (x-a)^\nu = \nu (x-a)^{\nu-1} = \frac{\mathrm{d}}{\mathrm{d}x}(x-a)^\nu.$$

*Also*

$$\lim_{\alpha\to 0} D_a^\alpha (x-a)^\nu = (x-a)^\nu.$$

*The fractional derivative $D^\alpha$ of order $\alpha \in (n-1, n]$ of the function $f(x)$ is given by*

$$D_a^\alpha f(x) = I_a^{n-\alpha} D^n f(x) \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}.$$

**Definition 1.4.** *Let $\alpha \in (n-1, n], n = 1, 2, \ldots$. The generalized Laguerre polynomials (generalized Rodrigues formula) $Ł_\alpha^\beta(x)$ of order $\alpha$ are defined by (see [13])*

$$Ł_\alpha^\beta(x) = \frac{x^{-\beta} \mathrm{e}^x}{\Gamma(1+\alpha)} D^\alpha \mathrm{e}^{-x} x^{\alpha+\beta} \qquad \beta > -1 \tag{4}$$

*and the generalized Laguerre polynomials $Ł_{-\alpha}^\beta(x)$ of order $-\alpha$ are defined by*

$$Ł_{-\alpha}^\beta(x) = \frac{x^{-\beta} \mathrm{e}^x}{\Gamma(1+\alpha)} I^\alpha \mathrm{e}^{-x} x^{-\alpha+\beta} \qquad \alpha + \beta > -1. \tag{5}$$

In [13], it is proved that $\{Ł_\alpha^\beta(x), \alpha \in \mathbb{R}\}$ are continuous as functions of $\alpha$, $\alpha \in \mathbb{R}$ and

$$\lim_{\alpha\to n^+} Ł_\alpha^\beta(x) = \lim_{\alpha\to n^-} Ł_\alpha^\beta(x) = L_n^\beta(x) \qquad n = 1, 2, 3, \ldots \tag{6}$$

the confluent hypergeometric representation of $Ł_\alpha^\beta(x), \alpha \in \mathbb{R}$ has been given and some other properties that generalize (interpolate) those of the classical Laguerre polynomials $L_n^\beta(x), n = 1, 2, \ldots$ have been proved.

Here we prove that $\{Ł_\alpha^\beta(x), \alpha \in \mathbb{R}\}$ are orthogonal in $L_2(0, \infty)$ with a weight function $\mathrm{e}^{-x/2}x^{\beta/2}$. Further, we prove that

$$\int_0^\infty \mathrm{e}^{-x} x^\beta \{Ł_\alpha^\beta(x)\}^2 \,\mathrm{d}x < \infty$$

and $Ł_\alpha^\beta(x)$ are particular solutions of the differential equation

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + (1+\beta-x)\frac{\mathrm{d}}{\mathrm{d}x}u(x) + \alpha u(x) = 0.$$

Also some applications in quantum mechanics are discussed.

## 2. Continuation properties

In [13], it is proved that (see corollary 4.1), if $\alpha \in (n-1, n]$, $n = 1, 2, 3, \ldots,$ then

$$\lim_{\alpha\to n} DŁ_\alpha^\beta(x) = DL_n^\beta(x).$$

In the same way we can prove the following two lemmas.

**Lemma 2.1.** *Let $\alpha \in (n-1, n]$, $n = 1, 2, \ldots$ and $\beta > -1$, then we have*

$$\lim_{\alpha\to n} D^m Ł_\alpha^\beta(x) = D^m L_n^\beta(x) \qquad m = 0, 1, 2, \ldots. \tag{7}$$

**Lemma 2.2.** *Let $\alpha \in (n-1, n]$, $n = 1, 2, \ldots$ and $\beta > \alpha - 1$, then we have*

$$\lim_{\alpha\to n} D^m Ł_{-\alpha}^\beta(x) = D^m L_{-n}^\beta(x) \qquad m = 0, 1, 2, \ldots. \tag{8}$$

## 3. The differential equation

Let $\alpha \in (n-1, n]$, $n = 1, 2, 3, \ldots,$ and $\beta \in \mathbb{R}$. Putting

$$Y_\alpha^\beta(x) = \frac{1}{\Gamma(1+\alpha+\beta)} D^\alpha \mathrm{e}^{-x} x^{\alpha+\beta} \qquad \beta > -1$$

then we have

**Theorem 3.1.** *The function $Y_\alpha^\beta(x)$ satisfies the following recurrence relations:*

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_\alpha^\beta(x) = \frac{\mathrm{d}}{\mathrm{d}x} Y_{\alpha-1}^\beta(x) - Y_\alpha^\beta(x) \tag{9}$$

$$(\alpha+\beta) Y_\alpha^\beta(x) = x\frac{\mathrm{d}}{\mathrm{d}x} Y_{\alpha-1}^\beta(x) + \alpha Y_{\alpha-1}^\beta(x) \tag{10}$$

$$(1+\alpha+\beta) Y_{\alpha+1}^\beta(x) = x\frac{\mathrm{d}}{\mathrm{d}x} Y_\alpha^\beta(x) + (1+\alpha) Y_\alpha^\beta(x). \tag{11}$$

**Proof.** From the definition of the function $Y_\alpha^\beta(x)$ and the properties of the fractional derivative we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}Y_\alpha^\beta(x) = \frac{1}{\Gamma(1+\alpha+\beta)}\frac{\mathrm{d}}{\mathrm{d}x}I^{n-\alpha}D^{n-1}\left[-\mathrm{e}^{-x}x^{\alpha+\beta} + (\alpha+\beta)\,\mathrm{e}^{-x}x^{\alpha+\beta-1}\right]$$
$$= \frac{-1}{\Gamma(1+\alpha+\beta)}D^\alpha \mathrm{e}^{-x}x^{\alpha+\beta} + \frac{\alpha+\beta}{\Gamma(1+\alpha+\beta)}\frac{\mathrm{d}}{\mathrm{d}x}D^{\alpha-1}\mathrm{e}^{-x}x^{\alpha-1+\beta} \tag{12}$$

from which the first result follows.

From the convergence of the power-series expansion of $\mathrm{e}^{-x}x^{\alpha+\beta}$ and the properties of the fractional derivative we obtain

$$Y_\alpha^\beta(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(1+m+\alpha+\beta)}{m!\Gamma(1+\alpha+\beta)\Gamma(1+m+\beta)}x^{m+\beta} \tag{13}$$

from which we can prove (by direct substitution) the second and third results. □

**Theorem 3.2.** *The function $Y_\alpha^\beta(x)$ is a particular solution of the differential equation*

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + (1+x-\beta)\frac{\mathrm{d}}{\mathrm{d}x}u(x) + (1+\alpha)\,u(x) = 0. \tag{14}$$

**Proof.** Substituting from (9) into (10) we obtain

$$(\alpha+\beta)Y_\alpha^\beta(x) = x\left\{\frac{\mathrm{d}}{\mathrm{d}x}Y_\alpha^\beta(x) + Y_\alpha^\beta(x)\right\} + \alpha Y_{\alpha-1}^\beta(x). \tag{15}$$

By differentiating (15) we obtain

$$(\alpha+\beta)\frac{\mathrm{d}}{\mathrm{d}x}Y_\alpha^\beta(x) - \frac{\mathrm{d}}{\mathrm{d}x}x\left\{\frac{\mathrm{d}}{\mathrm{d}x}Y_\alpha^\beta(x) + Y_\alpha^\beta(x)\right\} = \alpha\frac{\mathrm{d}}{\mathrm{d}x}Y_{\alpha-1}^\beta(x). \tag{16}$$

Again by substitution from (9) into the right-hand side of (16) we obtain

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}Y_\alpha^\beta(x) + (1+x-\beta)\frac{\mathrm{d}}{\mathrm{d}x}Y_\alpha^\beta(x) + (1+\alpha)Y_\alpha^\beta(x) = 0. \tag{17}$$

□

**Theorem 3.3.** *The generalized Laguerre polynomials $Ł_\alpha^\beta(x)$, $\alpha \in \mathbb{R}$ are particular solutions of the differential equation*

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + (1+\beta-x)\frac{\mathrm{d}}{\mathrm{d}x}u(x) + \alpha u(x) = 0. \tag{18}$$

**Proof.** First let $\alpha \in (n-1, n]$, $n = 1, 2, \ldots$. Since

$$Y_\alpha^\beta(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)}x^\beta \mathrm{e}^{-x}Ł_\alpha^\beta(x) \tag{19}$$

then substituting from (19) into (17) we obtain

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}Ł_\alpha^\beta(x) + (1+\beta-x)\frac{\mathrm{d}}{\mathrm{d}x}Ł_\alpha^\beta(x) + \alpha Ł_\alpha^\beta(x) = 0. \tag{20}$$

Secondly, since (see [13])

$$Ł_{-\alpha}^\beta(x) = \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)}{}_1F_1(\alpha, 1+\beta; x) \tag{21}$$

then by the direct calculation we obtain

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2}Ł_{-\alpha}^\beta(x) + (1+\beta-x)\frac{\mathrm{d}}{\mathrm{d}x}Ł_{-\alpha}^\beta(x) - \alpha Ł_{-\alpha}^\beta(x) = 0 \tag{22}$$

which completes the proof. □

## 4. Orthogonality property

**Theorem 4.1.** *For any two real numbers $\alpha_1 \neq \alpha_2$ we have*

$$\int_0^\infty \mathrm{e}^{-x} x^\beta \, Ł_{\alpha_1}^\beta(x) \, Ł_{\alpha_2}^\beta(x) \, \mathrm{d}x = 0. \tag{23}$$

**Proof.** Let $u_\alpha(x) = \mathrm{e}^{-x/2} x^{\beta/2} Ł_\alpha^\beta(x)$, $\alpha \in \mathbb{R}^+$, then by direct calculation we can prove that

$$x \frac{\mathrm{d}^2}{\mathrm{d}x^2} u_\alpha(x) + \frac{\mathrm{d}}{\mathrm{d}x} u_\alpha(x) + \left( \alpha - \frac{x}{4} + \frac{1+\beta}{2} - \frac{\beta^2}{4x} \right) u_\alpha(x) = 0. \tag{24}$$

Then for any two positive real numbers $\alpha_1 \neq \alpha_2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( x \frac{\mathrm{d}}{\mathrm{d}x} u_{\alpha_1}(x) \right) + \left( \alpha_1 - \frac{x}{4} + \frac{1+\beta}{2} - \frac{\beta^2}{4x} \right) u_{\alpha_1}(x) = 0 \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left( x \frac{\mathrm{d}}{\mathrm{d}x} u_{\alpha_2}(x) \right) + \left( \alpha_2 - \frac{x}{4} + \frac{1+\beta}{2} - \frac{\beta^2}{4x} \right) u_{\alpha_2}(x) = 0. \tag{26}$$

By multiplying (25) by $u_{\alpha_2}(x)$ and (26) by $u_{\alpha_1}(x)$, subtracting the resulting equations and integrating from 0 to $\infty$ we obtain

$$\int_0^\infty u_{\alpha_1}(x) \, u_{\alpha_2}(x) \, \mathrm{d}x = 0 \tag{27}$$

from which we obtain (23). □

Also, for any two negative real numbers $\alpha_1 \neq \alpha_2$, letting $u_{-\alpha}(x) = \mathrm{e}^{-x/2} x^{\beta/2} Ł_{-\alpha}^\beta(x)$, we can then prove that

$$x \frac{\mathrm{d}^2}{\mathrm{d}x^2} u_{-\alpha}(x) + \frac{\mathrm{d}}{\mathrm{d}x} u_{-\alpha}(x) + \left( -\alpha - \frac{x}{4} + \frac{1+\beta}{2} - \frac{\beta^2}{4x} \right) u_\alpha(x) = 0 \tag{28}$$

and

$$\int_0^\infty \mathrm{e}^{-x} x^\beta \, Ł_{-\alpha_1}^\beta(x) \, Ł_{-\alpha_2}^\beta(x) \, \mathrm{d}x = 0. \tag{29}$$

**Theorem 4.2.** *For any $\alpha \in \mathbb{R}$, $\beta > -1$, we have*

$$\int_0^\infty \mathrm{e}^{-x} x^\beta \left\{ Ł_\alpha^\beta(x) \right\}^2 \mathrm{d}x < \infty. \tag{30}$$

**Proof.** Let $\alpha \geqslant 0$, and adding (10) and (11) we obtain

$$(1+\alpha+\beta) Y_{\alpha+1}^\beta(x) - (1+2\alpha+\beta) Y_\alpha^\beta(x) = x \left( \frac{\mathrm{d}}{\mathrm{d}x} \left\{ Y_\alpha^\beta(x) - Y_{\alpha-1}^\beta(x) \right\} \right) - \alpha Y_{\alpha-1}^\beta(x) \tag{31}$$

substituting from (9) we obtain

$$(1+\alpha+\beta) Y_{\alpha+1}^\beta(x) - (1+2\alpha+\beta-x) Y_\alpha^\beta(x) + \alpha Y_{\alpha-1}^\beta(x) = 0. \tag{32}$$

Multiplying (32) by $\Gamma(2+\alpha+\beta)/\Gamma(1+\alpha)$, then from the definition of $Y_\alpha^\beta(x)$ and $Ł_\alpha^\beta(x)$ we can obtain

$$(1+\alpha) \, Ł_{\alpha+1}^\beta(x) - (1+2\alpha+\beta-x) \, Ł_\alpha^\beta(x) + (\alpha+\beta) \, Ł_{\alpha-1}^\beta(x) = 0 \tag{33}$$

which generalize the known formula for $L_n^\beta(x)$ (see [16]),

$$(1+n)L_{n+1}^\beta(x) - (1+2n+\beta-x)L_n^\beta(x) + (n+\beta)L_{n-1}^\beta(x) = 0. \tag{34}$$

Write (34) for $\alpha-1$ instead of $\alpha$, we obtain

$$\alpha Ł_\alpha^\beta(x) - (1+2\alpha-2+\beta-x)\,Ł_{\alpha-1}^\beta(x) + (\alpha-1+\beta)\,Ł_{\alpha-2}^\beta(x) = 0. \tag{35}$$

Now multiplying (33) by $Ł_{\alpha-1}^\beta(x)$ and (35) by $Ł_\alpha^\beta(x)$ and subtracting the two results we obtain

$$\begin{aligned}(1+\alpha)\,Ł_{\alpha+1}^\beta(x)\,Ł_{\alpha-1}^\beta(x) - \alpha\big(Ł_\alpha^\beta(x)\big)^2 - (1+\alpha+\beta)\,Ł_{\alpha-1}^\beta(x)\,Ł_\alpha^\beta(x)\\ +(\alpha+\beta)\big(Ł_{\alpha-1}^\beta(x)\big)^2 = 0.\end{aligned} \tag{36}$$

Multiplying by $\mathrm{e}^{-x}x^\beta$ and integrating from 0 to $\infty$ we obtain

$$\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x = \frac{(\alpha+\beta)}{\alpha}\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_{\alpha-1}^\beta(x)\big)^2\,\mathrm{d}x \tag{37}$$

from which we obtain

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x &= \frac{(\alpha+\beta)(\alpha+\beta-1)\ldots(\alpha+\beta-n+2)}{\alpha(\alpha-1)\ldots(\alpha-n+2)}\\ &\quad\times\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_{\alpha-n+1}^\beta(x)\big)^2\,\mathrm{d}x.\end{aligned} \tag{38}$$

Since $\alpha-n+1\in(0,1]$, let $\gamma=(\alpha-n+1]$ then the use of the relation (see [13], equation (39))

$$Ł_\gamma^\beta(x) = \frac{\Gamma(2-\gamma)(\gamma+\beta)}{\Gamma(1+\gamma)}Ł_{-(1-\gamma)}^\beta(x) - \frac{\Gamma(2-\gamma)x}{\Gamma(1+\gamma)}Ł_{-(1-\gamma)}^{\beta+1}(x) \tag{39}$$

and the hypergeometric representation of $Ł_{-\gamma}^\beta(x)$ we can obtain

$$Ł_\gamma^\beta(x) < \frac{\Gamma(2-\gamma)(\gamma+\beta)}{\Gamma(1+\gamma)}Ł_{-(1-\gamma)}^\beta(x) = \frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma)\Gamma(1+\beta)}\,{}_1F_1(1-\gamma,1+\beta;x).$$

Now since (see [19]) for $x\in[0,N]$, $N<\infty$, ${}_1F_1(1-\gamma,1+\beta;x)$ is bounded, $|{}_1F_1(1-\gamma,1+\beta;x)|<M$, then

$$\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_{\alpha-n+1}^\beta(x)\big)^2\,\mathrm{d}x < \frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma)\Gamma(1+\beta)}M^2\Gamma(1+\beta). \tag{40}$$

Hence,

$$\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x < M^2\frac{(\alpha+\beta)(\alpha+\beta-1)\ldots(\alpha+\beta-n+2)}{\alpha(\alpha-1)\ldots(\alpha-n+2)}\frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma)} \tag{41}$$

and

$$\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x < M^2\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} < \infty. \tag{42}$$

□

Also we have the following corollary which will be needed in the next paragraph.

**Corollary 4.1.** *For any* $\alpha\in\mathbb{R}$, $\beta>-1$ *we have*

$$\int_0^\infty \mathrm{e}^{-x}x^{\beta+1}\big\{Ł_\alpha^\beta(x)\big\}^2\,\mathrm{d}x < \infty. \tag{43}$$

**Proof.** Multiplying (33) by $Ł_\alpha^\beta(x)$ we obtain

$$(1+\alpha)\,Ł_\alpha^\beta(x)\,Ł_{\alpha+1}^\beta(x) - (1+2\alpha+\beta)\big(Ł_\alpha^\beta(x)\big)^2 + x\big(Ł_\alpha^\beta(x)\big)^2 + (\alpha+\beta)\,Ł_\alpha^\beta(x)\,Ł_{\alpha-1}^\beta(x) = 0. \tag{44}$$

By multiplying (44) by $\mathrm{e}^{-x}x^\beta$, integrating from 0 to $\infty$ and using (23) we obtain

$$0 - (1+2\alpha+\beta)\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x + \int_0^\infty \mathrm{e}^{-x}x^{\beta+1}\big(Ł_\alpha^\beta(x)\big)^2 + 0 = 0 \tag{45}$$

from which we obtain

$$\int_0^\infty \mathrm{e}^{-x}x^{\beta+1}\big(Ł_\alpha^\beta(x)\big)^2 = (1+2\alpha+\beta)\int_0^\infty \mathrm{e}^{-x}x^\beta\big(Ł_\alpha^\beta(x)\big)^2\,\mathrm{d}x < \infty. \tag{46}$$

□

## 5. Application

In quantum mechanics; the motion of a particle of mass $\mu$ in a field of central force whose potential is $V(r)$, and total energy $E$, is described by the Schrödinger equation for the wavefunction $\psi$

$$-\frac{h^2}{2\mu}\nabla^2\psi + V(r)\psi = E\psi \tag{47}$$

($h$ is Planck's constant divided by $2\pi$). The solution in the spherical polar coordinates $\psi(r,\theta,\phi)$ must satisfy the following conditions:

(a) $\psi(r,\theta,\phi) = \psi(r,\theta,\phi+2\pi)$
(b) $\psi$ is bounded for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant r < \infty$ and $0 \leqslant \phi < 2\pi$
(c) $\psi \to 0$ as $r \to \infty$
(d) $\psi$ is finite for $r \to 0$
(e) $\iiint |\psi|^2 r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = 1$.

The dependence on $\theta$ and $\phi$ which satisfies the above conditions can be written in the form

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(2ar)P_l^{|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi} \tag{48}$$

with $|m| \leqslant l$, $l = 0, 1, 2, \ldots$ and $a = \sqrt{-2\mu E/h^2}$ for negative energy.

For the hydrogen atom (see [17]) with the potential $V(r)$ equal to $-e^2/r$, we find that $R_{nl}$ is proportional to $\mathrm{e}^{-x/2}x^l L_{n-l-1}^{2l+1}(x)$, $n \geqslant l+1$ where

$$n = \frac{e^2}{h}\sqrt{\frac{\mu}{-2E}} \quad\Longrightarrow\quad E_n = -\frac{\mu e^4}{h^2(2n^2)}$$

and the condition ($e$) amounts to (compare with (46))

$$\int_0^\infty \mathrm{e}^{-x}x^{2l+2}\big(L_{n-l-1}^{2l+1}(x)\big)^2\,\mathrm{d}x = \frac{2n[(n+l+1)!]^3}{(n-l)!} < \infty. \tag{49}$$

Now we can say that the new definition of the Laguerre polynomials, the generalized Laguerre polynomials (generalized Rodrigues formula) of arbitrary (fractional) orders, enhances the field of the definition of the solutions for the hydrogen atom. This would add a continuous spectrum in between the discrete spectrum for the energies $E_n$. Further, this would open the question of the completeness of the solutions.

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